Time-duration extended Hilbert transform superposition for the reliable time domain analysis of five-layered damped sandwich beams

S.H. Bae a, J.R. Cho a,b,⁎, W.B. Jeong a

a School of Mechanical Engineering, Pusan National University, Busan 609-735, Republic of Korea
b Research & Development Institute of Midas IT, Gyeonggi 463-400, Republic of Korea

A R T I C L E   I N F O

Article history:
Received 11 November 2013
Received in revised form
6 June 2014
Accepted 12 June 2014

Keywords:
Viscoelastic sandwich beam
Time domain analysis
Time duration extension
Hilbert transform superposition
Newton-Raphson-based iterative residual method

A B S T R A C T

The direct time domain analysis of five-layered damped sandwich beams subject to impulse load using the discrete Hilbert transform may lead to the unstable growth in damped time responses owing to the incorrectness of Hilbert-transformed imaginary impulse force and the insufficient time duration of applied impulse force. To resolve this problem, a time domain analysis method making use of the Hilbert transform superposition and the time duration extension is introduced in this paper. The incorrect imaginary impulse force near the end of time period, which is caused by the fact that the discrete Hilbert transform using Fourier transform considers the impulse force as a periodic function, is resolved by dividing the external impulse force into a finite number of rectangular impulses. And, the sufficient time duration of applied impulse force is estimated by a Newton–Raphson-based iterative residual method. The validity of the present method is justified from the numerical experiment for analyzing the time response of five-layered damped sandwich beam.

© 2014 Elsevier B.V. All rights reserved.

1. Introduction

The damped vibration responses of complex structural dynamics such as sandwich beam structures with viscoelastic layers [1,2] are usually analyzed in either frequency or time domain. In the frequency domain analysis, the external excitation force is expressed as harmonic force so that the imaginary part is automatically defined. But, in the time domain analysis, the external excitation force is real contrary to the complex-valued dynamic equation system. In order to maintain the consistency in the complex-valued dynamic equation system, the real-valued external impulse force can be converted into an analytic force signal by defining the imaginary force signal using Hilbert transform [3–5]. Meanwhile, a state-space formulation in the modal superposition approach to solve the time response of the damped dynamic system leads to two poles which are radial symmetry in the complex plane [3]. The radial symmetry implies that one pole is stable but the other is unstable such that the standard numerical solving techniques cannot provide a successful solution because the unstable pole causes unbounded growth in the damped dynamic response. To avoid such an unbounded growth, the time-reversal technique [6,7] is used, in which the time differential equation corresponding to the unstable pole is converted to one running backwards in time.

However, the time domain analysis making use of the Hilbert transform and the time-reversal technique may lead to the incorrect damped time response owing to two problems, the incorrect Hilbert-transformed imaginary force and the insufficient time duration of applied external force. The Hilbert transform can be analytically derived according to its mathematical definition if the input force signal is expressed in a well-defined explicit function of time. However, for arbitrary input force signals, their Hilbert transforms are alternatively obtained by a combined use of Fourier (FT) and inverse Fourier transforms (IFT) according to the fundamental properties of the complex-valued strong analytical signal [5]. But, such a discrete Hilbert transform using FT and IFT may provide us the incorrect imaginary function near the end of time period because Fourier and inverse Fourier transforms consider the input force signal as a periodic function. Meanwhile, the time duration for the time domain analysis is set based on the original real-valued external force, but it may be insufficient for the imaginary external force. The Hilbert-transformed imaginary force may not be zero at the beginning even though its original real-valued external force starts with zero value, which gives rise to the incorrect unstable time response. This problem is solely owing to the fact that the time duration of complex-valued analytic force is set based on the original real-valued external force.

⁎ Corresponding author at: School of Mechanical Engineering, Pusan National University, Busan 609-735, Republic of Korea. Tel.: +82 51 510 3088; fax: +82 51 514 7640.
E-mail address: jrcho@pusan.ac.kr (J.R. Cho).
http://dx.doi.org/10.1016/j.finel.2014.06.007
0168-874X/© 2014 Elsevier B.V. All rights reserved.
force. It can be resolved by extending the time duration for the
time domain analysis so that the imaginary force starts with a
sufficiently small value.

As an extension of our work on the forced vibration of damped
sandwich beams [8,9], the purpose of this paper is to introduce a
time-duration extended Hilbert transform superposition method
for the reliable impulse response analysis of five-layered damped
sandwich beams in time domain. In order to resolve the above
mentioned problem of the discrete Hilbert transform by a com-
bined use of FT and IFT near the end of time period, the Hilbert
transform of arbitrary external force is obtained by the super-
position method. An arbitrary continuous real-valued force signal
is divided into a finite number of rectangular impulses and its
Hilbert transform is obtained by summing up the analytic Hilbert
transforms of each rectangular impulse. Since the analytic Hilbert
transform of rectangular impulse is derived by the mathematical
definition of Hilbert transform, the above-mentioned discrepance
near the end of time period can completely disappear. Meanwhile,
the original time duration set based on the real-valued external
force is extended by a Newton–Raphson-based iteration method,
based on the fact that Hilbert transform conserves the magnitude
of an original input signal [5]. The difference in squared areas
between the original and Hilbert-transformed functions is de-
ined as a residual functional, and the original time duration is success-
ively extended in both the negative and positive time directions
until the relative residual satisfies the preset tolerance.

The present method is applied to analyze the time response of
five-layered damped sandwich beam [10–12] in order to illustrate
and validate the theoretical work. The five-layered damped sand-
wich beam is modeled using 2-node 8-DOF damped beam elements
which are developed based on the principle of virtual work and the
compatibility relation between the lateral displacement and the
shear strains [9]. The complex finite element matrix equations are
derived in both time and frequency domain, and the time-
reversal technique is applied to the corresponding state-space
formulation in the mode superposition approach. Through the
numerical experiments, the validity of the Hilbert transform super-
position and the time duration extension algorithm are examined,
and the forced time response of five-layered damped sandwich
beam is compared with those obtained by IFT and the Hilbert
transform superposition method without extending the time
duration.

This paper is organized as follows. The finite element approx-
imation of the forced vibration of five-layered damped sandwich
beam using 8-DOF damped beam elements is addressed in Section 2.
The application of time-reversal technique for the time domain
analysis of linear complex structural dynamics is briefly summarized
in Section 3. The Hilbert transform superposition for defining the
correct imaginary impulse force and the time-duration extension
using a Newton–Raphson-based residual method are explained in
Section 4. The numerical experiments for illustrating the present
method are represented in Section 5 and the conclusion is made in
Section 6.

2. Forced vibration of five-layered damped sandwich beam

In the current study, we consider a five-layered damped sand-
wich beam shown in Fig. 1 as an engineering example of complex
structural dynamic problems. The beam of length $L$ and width $b$
with rectangular cross-section undergoes the lateral dynamic de-
celation under the action of time-varying vertical load $f(x, t)$. Three
metal layers of thicknesses $h_1, h_2$ and $h_3$ are linear elastic with
Young’s moduli of $E_1, E_2$ and $E_3$, while two darkened layers of
thicknesses $h_4$ and $h_5$ are linear viscoelastic with complex shear
moduli $C_{41} = G_{41}(1+i\eta_2)$ and $C_{54} = G_{54}(1+i\eta_4)$ respectively. Here, $\eta_2$ and $\eta_4$ indicate the loss factors and $(\cdot)^*$ denotes the complex value
throughout this paper. It is assumed that the beam thickness is
sufficiently small compared to the beam length and five layers are
completely bonded so as not to allow slipping at the layer interfaces.
Further assumptions are made for the analysis of the beam trans-
verse displacement: (1) the transverse direct strains in all the five
layers are small enough so that the lateral displacement is uniform
across any section of the beam, (2) the longitudinal and rotary
effects are negligible, (3) three metal layers obey Kirchhoff hypoth-
esis so as not to produce transverse shear strains, and (4) two
viscoelastic layers obey Kerwin assumption [13,14] so that the
longitudinal direct strains are much small than the shear strains.

The beam can be either symmetric or asymmetric, and the
transverse shear strains $\gamma_2$ and $\gamma_4$ in two viscoelastic layers are calculated by

$$\gamma_{il}^l = \frac{d_i}{h_i} \frac{\partial w(x, t)}{\partial x} + \frac{u_{l-1}(x) - u_{l+1}(x)}{h_l}, \quad l = 2, 4 \tag{1}$$
using the slopes of \( u_1, u_5 \) and \( u_5 \) with respect to the \( x \)-axis \([9,15–18]\). Then, the complex transverse shear stresses in two viscoelastic layers are calculated by \( \tau_{2}^{x} = G_{2}^{x} \gamma_{2}^{x} \) and \( \tau_{4}^{x} = G_{4}^{x} \gamma_{4}^{x} \). The transverse shear strains and stresses are uniform across the thickness because \( u_1, u_5, u_5 \) and \( w \) are independent of \( z \), which are consistent with the above mentioned assumption \([4]\). By denoting \( D_1 = E_1 h_1^2 / 12 \), \( D_3 = E_3 h_3^2 / 12 \) and \( D_5 = E_5 h_5^2 / 12 \) to be the flexural rigidities of three metal layers, the transverse shear resultants exerted on metal layers are calculated by

\[
S_j = D_k A_j \frac{\partial w}{\partial x^j} \quad j = 1, 3, 5
\]

Since the transverse shear resultants on two viscoelastic layers are \( S_2^x = -\tau_{2}^{x} A_2 \) and \( S_4^x = -\tau_{4}^{x} A_4 \), the total complex transverse shear force exerted on the sandwich beam cross-section becomes

\[
S^x = (DA) \frac{\partial w}{\partial x^j} - G_{2}^{x} \gamma_{2}^{x} - G_{4}^{x} \gamma_{4}^{x}
\]

with \((DA) = D_1 A_1 + D_3 A_3 + D_5 A_5\). Meanwhile, the axial resultant forces \( P_1, P_3 \) and \( P_5 \) in each metal layer are straightforwardly calculated by

\[
P_j = E_j A_j \frac{\partial u_j}{\partial x} \quad j = 1, 3, 5
\]

with \( A_j \) being the cross-section areas of each layer. The finite element approximation of forced vibration of five-layered sandwich beam is derived based on the principle of virtual work. The total kinetic energy \( T \) of the beam is given by

\[
T(t) = \int_{0}^{L} \rho \frac{1}{2} \left( \begin{array}{c}
\frac{\partial w}{\partial x} \frac{\partial w}{\partial x} \\
\frac{\partial u_1}{\partial x} \frac{\partial u_1}{\partial x} \\
\frac{\partial u_3}{\partial x} \frac{\partial u_3}{\partial x} \\
\frac{\partial u_5}{\partial x} \frac{\partial u_5}{\partial x}
\end{array} \right) \text{d}x
\]

and the transverse shear strain energy \( V_1 \) of two viscoelastic layers is

\[
V_1(t) = \int_{0}^{L} \left[ E_1 A_1 \left( \frac{\partial u_1}{\partial x} \right)^2 + E_3 A_3 \left( \frac{\partial u_3}{\partial x} \right)^2 + E_5 A_5 \left( \frac{\partial u_5}{\partial x} \right)^2 \right] \text{d}x
\]

\[
= \frac{Z_1}{2} \int_{0}^{L} \left[ (h_1 \gamma_2^x - d_2 w^x)^2 - Z_3(h_4 \gamma_4^x - d_4 w^x)^2 + Z_5(h_2 \gamma_2^x + h_4 \gamma_4^x - d_2 w^x)^2 \right] \text{d}x
\]

and the transverse shear strain energy \( V_2 \) of two viscoelastic layers is

\[
V_2(t) = \frac{1}{2} \int_{0}^{L} \left( G_{2}^{x} \gamma_{2}^{x} + G_{4}^{x} \gamma_{4}^{x} \right)^2 \text{d}x
\]

with \( Z_1 = E_1 E_3 A_1 A_3 / (E_1 A_1 + E_3 A_3 + E_5 A_5) \) and \( Z_3 = E_3 A_3 / (E_1 A_1 + E_3 A_3) \) and \( Z_5 = E_5 A_5 / E_3 A_3 A_1 \). Note that \( u_1, u_3 \) and \( u_5 \) in Eq. (7) are replaced with \( w, \gamma_2^x \) and \( \gamma_4^x \) using Eqs. (1) and (2) and the constraint \([9]\): \( E_1 A_1 u_1 + E_3 A_3 u_3 + E_5 A_5 u_5 = 0 \).

The lateral deflection \( w \) and two transverse shear strains \( \gamma_2^x \) and \( \gamma_4^x \) are interpolated using 2-node damped sandwich beam shown in Fig. 2, where beam elements of length \( 2a \) which are discretized for the damped sandwich structure are mapped onto the master element with the length of 2. The interpolation of three fields \( w, \gamma_2^x \) and \( \gamma_4^x \) within each damped beam element of length \( 2a \) is given by

\[
w(\xi) = [N^w(\xi)][w], \quad \gamma_2^x(\xi) = [N^{\gamma_2^x}(\xi)][\gamma_2^x], \quad \gamma_4^x(\xi) = [N^{\gamma_4^x}(\xi)][\gamma_4^x]
\]

with \([w] = [w \quad \gamma_2^x \quad \gamma_4^x] \) being the element-wise nodal vector. Note that the rotation \( \theta \) of beam axis is added to implement all the possible essential boundary conditions for damped sandwich beams \([9,16]\), and it is explicitly interpolated using the relation of \( \theta = \omega w / \partial \xi \). Each complex matrix \([N^w(\xi)], [N^{\gamma_2^x}(\xi)] \) and \([N^{\gamma_4^x}(\xi)]\) is composed of eight complex shape functions \( N^w(\xi), N^{\gamma_2^x}(\xi) \) and \( N^{\gamma_4^x}(\xi) \) respectively, which are in function of the loss factor \( \eta \), the material properties and geometry dimensions of the damped sandwich beam. The detailed derivation of these complex shape functions is out of scope of this paper, but the reader may refer to our previous paper \([9]\) for the concept of polynomial-based damped sandwich beam.

Substituting \( w(\xi,t) \) from Eq. (9) into Eq. (5) leads to the element-wise complex mass matrices defined by

\[
[M]^e = a \int_{-1}^{1} \rho [N^e]^T [N^e] d^e \xi
\]

and, substituting Eq. (9) into Eqs. (6)-(8) provides us the element-wise complex stiffness matrices given by

\[
[K]^e = (DA) \int_{-1}^{1} [N^{e*}]^T [N^e] d^e \xi + a \int_{-1}^{1} \left[ G_2^{e} A_2 [H^e]^T [H^e] + G_4^{e} A_4 [I^e]^T [I^e] \right] d^e \xi + d_2 \int_{-1}^{1} \Lambda^e(\xi) d^e \xi
\]

where \( \Lambda^e(\xi) \) indicates the integrand stemming from the extensional strain energy of three metal layers. Meanwhile, the element-wise load vectors are defined by

\[
[f]^e = a \int_{-1}^{1} f(\xi,t) [N^e]^T d^e \xi
\]

Summing up the element-wise matrices and vectors over all the elements leads to a complex linear matrix equation system of motion:

\[
[M]^T [\ddot{w}] + [K]^e [\dot{w}] = [f]^e
\]

Differing from the frequency domain analysis where the external excitation force is expressed as a harmonic function, the real-valued external impulse force \( [f] \) in the time domain analysis should be converted into a complex-valued analytic force \( [f^*] \) in order for the consistency of the complex-valued dynamic equation system (13). One may obtain the impulse response, such as the peak dynamic displacement and the decaying characteristics, by taking the inverse Fourier transform (IFT) of the frequency response. But, not only such Fourier and inverse Fourier transforms become an extra job but also the converted impulse response in time domain is affected by the resolution of frequency response function (FRF). In this context, a somewhat novel direct time-domain analysis method for Eq. (13) is introduced in this paper, motivated by the fact that an arbitrary real-valued impulse
force can be decomposed into a finite number of well-defined rectangular impulse signals and its imaginary part can be obtained by superposing the Hilbert transforms of those rectangular impulses. Furthermore, the resulting complex-valued analytic impulse force could provide the accurate impulse response when its time duration is appropriately extended.

3. Time domain analysis

Let \( w_a(t) \) and \( f_a(t) \) be the analytic signals (i.e., complex signals composed of real and imaginary parts) of the damped beam deflection \( w(t) \) and the external excitation force \( f(t) \), the previous linear complex equation system (13) for the forced vibration of five-layered damped sandwich beam is rewritten as

\[
[M^a][\ddot{w}_a]+[K^a][\dot{w}_a]=[f_a^a]
\]

(14)

The complex mass and stiffness matrices \([M^a]\) and \([K^a]\) which are expressed in terms of the complex shear modulus \(G^a\) are not Hermitian but symmetrically similar to those for the finite element approximation of linear elasticity problems. And, the complex matrix Eq. (14) of motion leads to the conjugate complex eigenvalues, which causes the numerical instability and divergence when the time response is sought by the standard time integration schemes like the discrete-time implementation of the convolution signal [3]. In order to overcome such a numerical instability and divergence, the time-reversal technique which has been studied for a long time in acoustic and medical applications [6,7] is widely adopted.

In order for the time-domain analysis by the mode superposition approach, the analytic deflection \( w_a(t) \) is expanded as a linear combination of the analytic participation coefficients \( p_a^n(t) \) and the complex natural modes \([X^a_n]\) such that

\[
w_a(t) = [p_a(t)]^T[X^a]
\]

(15)

Substituting \( w_a(t) \) into the above complex matrix Eq. (14) leads to the decoupled complex ordinary differential equations for each analytic participation coefficient \( p_a^n(t) \) [17,18]:

\[
m_a^n p_a^n + \alpha_a^n (1+i\eta_p) p_a^n = (X^a_n)^T[f_a(t)]
\]

(16)

with \( m_a^n = (X^a_n)^T[M^a][X^a_n] \) being the \( M \)-orthogonalized complex-valued mass. A state-space formulation to solve the time response of Eq. (16) becomes

\[
[y_a(t)] = [A][y_a(t)] + [B][f_a(t)], \quad [y_a(t)] = \begin{bmatrix} p_a(t) \\ \dot{p}_a(t) \end{bmatrix}
\]

(17)

with two matrices defined by

\[
[A] = \begin{bmatrix} 0 & 1 \\ -\alpha_a^n (1+i\eta_p) & 0 \end{bmatrix}, \quad [B] = \begin{bmatrix} 0 \\ (X^a_n)^T/m_a^n \end{bmatrix}
\]

(18)

Let us denote \((s^a_1, s^a_2)\) and \((\Phi_1, \Phi_2)\) to be the complex eigenvalues and eigenvectors of \([A]\), respectively, where \(s^a_1 = -\alpha + i\eta_p\) and \(s^a_2 = -\alpha - i\eta_p\). Similarly, let \((s^a_3, s^a_4)\) be \((s^a_1, s^a_2)\) but \(s^a_3 = \alpha + i\eta_p \) and \(s^a_4 = \alpha - i\eta_p\) for the sake of simplicity. Referring to Inaudi and Makis [3], \(-s^a_2\) causes the unbounded growth in the time response \(y_a(t)\) of the hysteretic damping system when a standard time integration scheme is used.

In order to uncouple the dynamics of the bounded response by \(s^a_2\) from the unbounded one by \(-s^a_2\), \(y_a(t)\) is expressed in terms of the eigenvectors of \([A]\) by

\[
[y_a(t)] = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}, \quad [\Phi] = \{\Phi_1 \Phi_2\} = \begin{bmatrix} 1 & 1 \\ s^a_2 & -s^a_2 \end{bmatrix}
\]

(19)

by defining \(q_1(t)\) and \(q_2(t)\) be the analytic modal coordinates. Reversing the time in the modal coordinate \(q_2(t)\) as \(t = -t\) and \(\dot{q}_2(t) = q_2(t)\), the time derivative of \(q_2(t)\) is transformed into

\[
\dot{q}_2(t) = \frac{dq_2(t)}{dt} = \frac{d\dot{q}_2(t)}{d\dot{t}} = -\dot{q}_2(t)
\]

(20)

Then, the previous state-space formulation (17) can be rewritten as

\[
[\Phi] \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = [A][\Phi] \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + [B][\ddot{f}_a(t)]
\]

(21)

Using the relation given by \([\Phi]^T[A][\Phi] = \text{diag}(s^a_1^2, -s^a_2^2)\), one can obtain two stable differential equations given by

\[
\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} s^a_1^2 & 0 \\ 0 & -s^a_2^2 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 1/2s^a_1^2 m_a^n \\ -1/2s^a_2^2 m_a^n \end{bmatrix}[X^a_n]^T[\ddot{f}_a(t)]
\]

(22)

in which the first one runs in forward in \(t\) while the second one runs in backward in \(t\).

4. Time duration extended Hilbert transform superposition

4.1. Hilbert transform superposition

The Hilbert transform \(H[f(t)]\) of a signal \(f(t)\) is a linear operator defined by the convolution of \(f(t)\) and such that

\[
\hat{f}(t) = H[f(t)] = \frac{1}{\pi} \lim_{\tau \to \infty} \int_{-\tau}^{\tau} \frac{-f(t)}{t-t'} dt
\]

(23)

with the Cauchy principal value given by

\[
P \int_{-\tau}^{\tau} \frac{f(t)}{t-t'} dt = \lim_{\varepsilon \to 0} \left[ \int_{-\tau-\varepsilon}^{\tau+\varepsilon} f(t) \frac{dt}{t-t'} + \int_{\tau+\varepsilon}^{\tau-\varepsilon} f(t) \frac{dt}{t-t'} \right]
\]

(24)

around \(\tau\) = \(t\). The Hilbert transform conserves the magnitude of an original signal \(f(t)\) and only produces a phase shift, and furthermore it possesses a linearity property such that \(a\hat{f}(t)+b\hat{g}(t)\) is the Hilbert transform of \(f(\alpha t)+g(\beta t)\) for every \(\alpha, \beta > 0\) [3,5].

From the fact that the Hilbert transform is defined by the convolution \(\hat{f}(t) = f(t)\ast(-1/\pi t)\) and the Fourier transform (FT) of \((-1/\pi t)\) becomes

\[
\text{FT}[\hat{f}(\omega)] = i \text{sgn}(\omega)F(\omega)
\]

(25)

with \(\text{sgn}(\omega) = -1\) for \(\omega > 0\) and \(0\) for \(\omega = 0\) and 1 for \(\omega < 0\), the Fourier transform of the Hilbert transform \(f(t)\) can be obtained using the Fourier transform \(F(\omega)\) of the original signal \(f(t)\) such that

\[
\text{FT}[\hat{f}(\omega)] = i \text{sgn}(\omega)F(\omega)
\]

(26)

Thus, the Fourier transform of a strong analytic signal \(f_a(t) = f(t) + \hat{i}f(t)\) is defined by

\[
\text{FT}[f_a(t)] = F_a(\omega) = F(\omega) + i\text{sgn}(\omega)F(\omega)
\]

(27)

and, thus the strong analytic signal \(f_a(t) = f(t) + \hat{i}f(t)\) can be obtained by the inverse Fourier transform (IFT) of the one-side-spectrum \(F^+(\omega) = F(\omega), \omega > 0\):

\[
\text{IFT}[F^+(\omega)] = f_a(t)
\]

(28)

Thus, the analytic signal of the real applied load for the forced vibration analysis of damped sandwich beam would be obtained by a combined use of discrete FT and IFT.

But, the imaginary part \(\hat{f}(t)\) in analytic signal which is obtained using Eq. (28) may be different from one derived analytically using the definition of Hilbert transform (23) because the Fourier transform regards the input signal as a periodic function. In order to investigate this difference, let us consider a rectangular impulse \(f(t) = \text{rect}(t)\) shown in Fig. 3(a). It is not hard to derive the Hilbert
transform \( \hat{f}(t) \) given by

\[
H[\text{rect}(t)] = \hat{f}(t) = \frac{1}{\pi} \ln|t - t_1|/|t - t_2|
\]

by substituting \( \text{rect}(t) \) into Eq. (23), and the Hilbert transform for \( t_1 = 0 \) s and \( t_2 = 1.0 \) s is represented in Fig. 3(b). Where the other two Hilbert transforms are obtained using Eq. (28), one for a non-periodic single rectangular signal as shown in Fig. 3(a) and the other considering the rectangular signal as a periodic function.

It is clearly observed that the Hilbert transform obtained using the discrete FT and IFT shows the totally different curve from the analytically derived one near the end of time period. But, the Hilbert transform obtained by the discrete FT and IFT exactly coincides with one of the periodic rectangular signal, justifying that non-periodic signals are regarded as a periodic one when their analytic signals are numerically obtained by FT and IFT. Therefore, the numerical approach to obtain the analytic signal \( f_a(t) \) using Eq. (28) cannot be used without extra numerical treatment, because the numerically obtained Hilbert transform showing the discrepant signal near the end of time period leads to the incorrect time response for the hysteresis damping system. In the current study, a Hilbert transform superposition method is introduced to overcome this problem. This method is motivated by the fact that the analytically derived Hilbert transform does not produce the above-mentioned problem and an arbitrary impulse signal can be decomposed into a finite number of well-defined impulse signals.

Referring to Fig. 4, let us denote \( D(t) \) be the discrete impulse function of \( f(t) \) which is defined by

\[
D(t) = \sum_{k=1}^{n-1} D_k(t), \quad D_k(t) = h_k(t_k \leq t \leq t_{k+1})
\]

using a finite number of rectangular impulses \( D_k(t) \). Substituting \( D(t) \) into Eq. (23), one can easily derive the Hilbert transform \( H[D(t)] \) given by

\[
H[D(t)] = \frac{1}{\pi} \sum_{i=1}^{n} d_i \ln|t - t_i|
\]

where \( d_i = h_1, d_n = -h_{n-1} \) and \( d_i = h_i - h_{i-1} (i = 2, 3, \ldots, n-1) \). Note that the value of \( t \) at \( t_i \) is replaced with the mean of \( t_i \) and \( t_{i+1} \) in order to prevent \( \ln|t - t_i| \) from becoming infinity.

Fig. 5 shows the Hilbert transforms of rectangular and triangular impulse signals using the Hilbert transform superposition with \( n = 10000 \) for rectangular impulse and \( n = 1000 \) for triangular impulse, where the above mentioned problem shown in Fig. 3 of the conventional Hilbert transform near the end of time period completely disappears. For the current study, thanks to this feature of the Hilbert transform superposition, the analytic signal \( f_a(t) \) of real applied force \( f(t) \) is generated by combining the original real-valued applied force \( f(t) \) and its imaginary-valued transformed signal \( \hat{f}(t) \) which is obtained by the Hilbert transform superposition.

4.2 Estimation of time duration extension

Another point to be considered for Hilbert transform is that the imaginary part \( \hat{f}(t) \) may not be zero even at the time when the real impulse signal \( f(t) \) starts with zero value, as illustrated in Fig. 5 (a) and (b). There is no doubt that such an analytic impulse signal \( f_a(t) \) leads to the incorrect damped time response of structural dynamic system, so the time duration of analytic impulse force \( f_a(t) \) which is based on the real part \( f(t) \) should be adjusted. In other words, the time duration of analytic impulse force should be extended so that the imaginary part \( \hat{f}(t) \) reasonably approaches zero value. Referring to Fig. 6, the time duration \( [t_0^L, t_n^L] \) which is set based on the real part \( f(t) \) should be extended in both the negative and positive time directions, where the time duration extension is made by the Newton–Raphson iteration based on the fact that Hilbert transform conserves the magnitude of an original real impulse signal.

The time duration is extended equally in the negative and positive time directions, and let \( [t_{i+1}^L, t_{i+1}^R] \) be the time duration to be determined at the next Newton–Raphson iteration \( i + 1 \). Then, it can be expressed in an iterative linearized form given by

\[
t_{i+1}^L = t_i^L - \frac{\hat{f}(t_i^L)}{\hat{f}'(t_i^L)}, \quad i = 0, 1, 2, \ldots \quad (L = L, R)
\]
with \( \dot{f}(t) \) being the slope of imaginary impulse force at time \( t \).

In addition, let us introduce the residual \( R_e \) defined by

\[
R_e = \text{IN}(f(t)) - \text{IN}(H(f(t)))
\]

at iteration stage \( i \), where

\[
\text{IN}(f(t)) = \int_{t_i}^{t_f} |f(t)|^2 dt, \quad \text{IN}(H(f(t))) = \int_{t_i}^{t_f} |H(f(t))|^2 dt
\]

(35)

The residual means the difference in squared area between the original real function \( f(t) \) and the transformed imaginary one \( H(f(t)) \).

The flowchart for the iterative calculation of the extended time duration is represented in Fig. 7, where the termination of iteration is judged by two values, \(|df/dt|\) and \(|Re/Re_{i-1}|\). The convergence tolerance \( \epsilon_f \) is successively reduced from 1.0 until another tolerance \( \epsilon_r \) is satisfied.

5. Numerical experiment

The numerical method for estimating the extend time duration is applied to the rectangular impulse shown in Fig. 5(a) with the time duration \( t \in [0, 2] \). By adjusting the slope \(|df/dt|\) from 1.0 to 0.0001, the extended time duration, the total number of iterations, \( Re/\text{IN}(H(f)) \) and \( Re_{i+1}/Re_i \) are investigated, as given in Table 1. It is observed that the relative residual \( Re/\text{IN}(H(f)) \) and the residual ratio \( Re_{i+1}/Re_i \) show uniform decrease and uniform increase and approach 2.0% and 100%, respectively. On the other hand, the total number of iterations and the extended time duration dramatically increase in proportional to \(|df/dt|\). Owing to the asymmetric distribution of \( f(t) \) with respect to the time axis, the total numbers of iterations in the negative and positive time directions are different.

The saturation trend of \( Re/\text{IN}(H(f)) \) to \(|df/dt|\) is clearly observed in Fig. 8. There is no doubt that the analytic external force with longer extended time duration provides us more accurate time response of complex structural dynamic problem, but the total CPU time required for the time domain analysis becomes a burden. In this context, the choice of appropriate extended time duration should be made by considering these two conflicting aspects. Restricted to the convergence characteristic of the relative residual \( Re/\text{IN}(H(f)) \), the tolerances \( \epsilon_f \) for \(|df/dt|\) and \( \epsilon_r \) for \( Re_{i+1}/Re_i \) are suggested to be lower than 0.001 and greater than 98%, respectively.

We next apply the proposed time-duration extended Hilbert transform superposition to five-layered damped sandwich beam to analyze the time response to impulse force. Fig. 8(b) shows a fixed–fixed asymmetric five-layered damped sandwich beam subject to a double half sine impulse shown in Fig. 9(a) at the center. The material properties of aluminum and rubber layers are given in Table 2, and the sandwich beam is uniformly discretized with 17 damped beam elements presented in Section 2. The loss factor \( \eta \) for rubber layers is set by 0.2, and the damped time response of the beam is taken at the point where the impulse force is applied.
Fig. 9(a) represents the Hilbert transform of double half sine by the superposition method explained in Section 4.1 before the extension of time duration \( t = 0 \) s, and this sudden non-vanishing imaginary force at the time duration becomes 0.001 s. It is observed that the discrete Hilbert transform far from the exact one when the shape of external impulse load is not rectangular but arbitrary. In order to investigate the influence of this subject on the present time analysis method, let us consider a double triangle impulse load shown in Fig. 11(a). Referring to the previous Fig. 4, let us discretize the impulse load in two different ways, one is the present division into a finite number of rectangular impulses and the other is the piecewise linear interpolation. In the latter case, the discrete Hilbert transform \( H[D(t)] \) is derived as

\[
H[D(t)] = h_1 - h_0 + \frac{1}{\pi} \sum_{i=1}^{n} d(t_i) \ln|t - t_i|
\]  

(36)

with \( d(t_i) \) being the \( i \)th piecewise linear function between \( t_{i-1} \) and \( t_i \). One can realize that Eq. (36) is different from Eq. (32) for the case when the external impulse load is divided into rectangular impulses. Fig. 11(b) comparatively represents the discrete Hilbert transforms of the double triangle impulse load when the time interval \( dt \) is set by 0.01 s. It is observed that the discrete Hilbert transform obtained by the linear interpolation method is much closer to the exact one. But, the difference between two
methods significantly diminishes as the time interval becomes smaller, such that the transformed signals obtained by both methods are in perfect agreement with the exact one when $dt$ reaches 0.0001 s, as shown in Fig. 12.

Next, the difference in the time responses of damped sandwich beam between the present rectangular impulse superposition and the piecewise linear interpolation is investigated. The fixed–fixed five-layered damped sandwich beam shown in Fig. 8(b) is also taken, where the loss factor $\eta$ and the thicknesses of five layers are changed to 0.1 and 10.5 mm (1, 1, 6, 2 and 0.5 mm from the top layer) respectively. The time duration of impulse load is extended because the transformed imaginary impulse signals start with non-vanishing negative values at $t = 0$ s. Two tolerances $\varepsilon_f$ and $\varepsilon_t$ for $|df/dt|$ and $Re_{i+1}/Re_i$ required for the iterative extension of time duration are set by 0.001 and 99.9%, respectively, as in the previous case for the double half sine impulse. The transformed impulse signals shown in Fig. 12 which were obtained with $dt = 0.001$ and 0.0001 s are taken for the time response analyses, because the time intervals larger than 0.001 s produce the discrete Hilbert transforms far from the exact one. For the both time intervals, the total iteration numbers for the time duration extension were 14 for the present rectangular impulse superposition and 21 for the piecewise linear interpolation. Meanwhile, for both signal discretization methods, the extended time durations are as follows: $149 \times 140$ for $dt = 0.001$ s and $113 \times 298$ for $dt = 0.0001$ s. Thus, it has been observed that the total iteration number becomes larger for the piecewise linear interpolation and the extended time duration increases as the time interval becomes smaller.

The damped time responses of beam deflection at the point where the impulse force is applied are compared in Fig. 13, where the extended time durations were adjusted so that the time response starts from $t = 0$ s. The difference in the time responses between two methods is observed at $dt = 0.001$ s, particularly in

Fig. 10. Comparison of time responses between the present method and IFT: (a) overall and (b) detailed.

Fig. 11. (a) Double triangle impulse and (b) comparison of Hilbert transforms ($dt = 0.01$ s).

Fig. 12. Dependence of Hilbert transform on the time interval $dt$: (a) $dt = 0.001$ s and (b) $dt = 0.0001$ s.
the amplitudes, but it almost completely disappears when the transformed signals obtained using $dt$ of 0.0001 s are applied. Thus, the present method using the rectangular impulse superposition can not only produce the consistent discrete Hilbert transform for non-constant arbitrary impulse forces, but it can also lead to the accurate time response of the damped sandwich beam when the time discretization interval is sufficiently small.

6. Conclusion

In this paper, a time-duration extended Hilbert transform superposition method has been introduced in order for the reliable direct time domain analysis of five-layered damped sandwich beams subject to impulse load. The conventional discrete Hilbert transform using FT and IFT which consider the input impulse forces as a periodic function leads to the incorrect Hilbert transform near the end of time period. Meanwhile, the Hilbert-transformed imaginary part of the impulse force may not be zero even at the time when the real impulse signal starts with zero value. These problems of conventional discrete Hilbert transform using FT and IFT for the direct time-domain analysis substantially lead to the unstable damped time response, and such problems have been resolved by introducing the Hilbert transform superposition and the time duration extension. The external impulse force was divided into a finite number of rectangular impulses and its Hilbert transform was obtained by superposing Hilbert transforms of each rectangular impulse. The original time duration set based on the real part of impulse force has been extended by a Newton–Raphson-based iterative residual method so that the non-vanishing imaginary impulse force at the beginning does not cause the unstable growth. From the numerical experiment for the time response analysis of five-layered damped sandwich beam, it has been justified that the present method successfully provides us a stable decaying time response which is consistent with the solution obtained by IFT. In addition, from the comparison with the piecewise linear interpolation method for discretizing the external impulse load, it has been justified that the present method can not only produce the consistent discrete Hilbert transform for non-constant arbitrary impulse forces, but it can also lead to the accurate time response, provided that the external impulse load is discretized by a sufficiently small time interval.

Acknowledgment

This work was supported by the Human Resources Development of the Korea Institute of Energy Technology Evaluation and Planning (KETEP) grant funded by the Korea Ministry of Knowledge Economy (No. 201103020020010). The financial support for this work through World Class 300 from Ministry of Knowledge and Economy of Korea is acknowledged.

References